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Quantum Shot Noise: Expansions in Powers of the Pulse Density

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Quantum shot noise consists of individual pulses which contribute time-dependent (operator) "potentials" toward a total potential $V(t)$. The averaged quantity $\langle \mathscr{T} \exp \int_{t_0}^{t} dt' \, V(t') \rangle$ in general can no longer be calculated explicitly, in contrast to the classical case, and expansions are of interest. Noncommutative cumulant expansions are not directly applicable if the correlation functions of $V(t)$ have singularities, as happens in applications. It is shown here that these expansions, when applied to quantum shot noise, can be partially summed to give expansions in powers of the pulse density ν. Three types of such expansions are established explicitly, and for two of them the derivation is direct. For one of them the first-order approximation is closely connected to the so-called unified theory of spectral-line broadening.

KEY WORDS: Operator Poisson process; singular correlation functions; noncommutative cumulant expansions; partial summation.

1. INTRODUCTION AND MAIN RESULTS

The familiar classical shot noise,[2] denoted by $S(t)$, is a sum of scalar pulses $h(t-\tau_k)$ occurring with density ν on the time axis,

$$S(t) = \sum_k h(t-\tau_k)$$

In quantum situations, on the other hand, single pulses contribute operator functions ("potentials"), which usually do not commute for different times. A simple example of such a situation can be given by a hydrogen atom at the origin and a narrow beam of classical particles passing the atom at a fixed distance; each particle is assumed to contribute a potential pulse to

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the total potential of the electron, and in general these pulses need not commute in the interaction picture.[3] A generalization of this example would be a hydrogen atom in a plasma, and this is just the situation studied in the theory of spectral-line broadening. This is no longer pure shot noise, since each pulse depends on additional parameters. Our results on the pure case can, however, be applied to spectral-line broadening, as explained elsewhere.[4]

In the quantum case we denote by $V(t)$ the sum of all single-pulse contributions. Of interest in many applications is the quantity

$$\langle U(t, t_0) \rangle \equiv \left\langle \mathscr{T} \exp\left[\int_{t_0}^{t} dt'\, V(t')\right]\right\rangle \tag{1.1}$$

which is a generalization of the well-known random frequency modulation.[5] Note that $\langle U(t, t_0) \rangle$ is the averaged solution of the stochastic differential equation

$$\dot{U} = VU \tag{1.2}$$

In the scalar case this can be calculated explicitly in terms of single-pulse quantities.[2] In the quantum case this is in general no longer possible, and we therefore investigate expansions in the pulse density ν. To do this, one might try to use our companion paper[1] (hereafter referred to as I), where different types of noncommutative cumulant expansions for stochastic differential equations as in Eq. (1.2) were investigated. However, depending on the particular form of the single pulse shape, there may be singularities in the correlation functions of V, as happens in the theory of spectral-line broadening. Then the cumulant expansions of Eqs. (I.1.5)–(I.1.7) are not applicable, at least not directly. Surprisingly, though, it turns out that for quantum shot noise these cumulant expansions can be *partially summed* and the singularities then disappear. This will now be explained in more detail.

1.1. Quantum Shot Noise

We consider "particles" arriving randomly with average pulse density ν at some location. A particle arriving at time τ contributes a time-dependent matrix or operator "potential" $\varphi(t; \tau)$, a single "pulse." The total "potential"—possibly nonstationary—is then given by

$$V(t) = \sum_k \varphi(t; \tau_k)$$

where a possible factor of i in Eq. (1.2) has been absorbed in φ. As usual, the arrival times τ_k are assumed to be uniformly distributed either over a finite time interval of length T or, in the limit, over the whole real axis. We then have, with $N/T = \nu$,

$$\begin{aligned}\langle V(t)\rangle &= \lim_{N\to\infty} \sum_1^N \int_{-T/2}^{T/2} \frac{d\tau_k}{T}\, \varphi(t;\tau_k) \\ &= \nu \int d\tau\, \varphi(t;\tau) \end{aligned} \tag{1.3}$$

In a similar way one obtains

$$\begin{aligned}\langle V(t_1)\, V(t_2)\rangle &= \nu \int d\tau\, \varphi(t_1;\tau)\, \varphi(t_2;\tau) \\ &\quad + \nu^2 \int d\tau_1\, \varphi(t_1;\tau_1) \int d\tau_2\, \varphi(t_2;\tau_2) \end{aligned} \tag{1.4}$$

where the first term arises from the summation over equal pulses, as explained in the Appendix. It is similarly shown there that the correlation function $\langle V(t_1)\cdots V(t_n)\rangle$ is a polynomial in ν of nth degree starting with

$$\nu \int d\tau\, \varphi(t_1;\tau)\cdots \varphi(t_n;\tau) \tag{1.5}$$

It is clear that, depending on the behavior of $\varphi(t;\tau)$ as a function of τ, e.g., for $|\tau| \to \infty$, there may be singularities in specific correlation functions. In such a case we assume φ to be replaced by a regularized version and then remove the regularization in the final expressions.

1.2. Density Expansion through an Integral Equation

For a general stochastic differential equation as in Eq. (1.2), an integral equation for $\langle U(t, t_0)\rangle$ was studied in I,

$$\langle U(t, t_0)\rangle = 1 + \int_{t_0}^{t} ds\, G_{\mathrm{I}}(t, s)\langle U(s, t_0)\rangle \tag{1.6}$$

The kernel G_{I} is given in terms of W-cumulants by Eq. (I.1.5),

$$\begin{aligned} G_{\mathrm{I}}(t, s) &= \left\langle \mathscr{T} \exp\left[\int_s^t ds'\, V(s')\right] V(s) \right\rangle^{\mathrm{W}} \\ &= \sum_0^\infty \int_s^t ds_1 \cdots \int_s^{t_{n-1}} dt_n\, \langle V(t_1)\cdots V(t_n)\, V(s)\rangle^{\mathrm{W}} \end{aligned} \tag{1.7}$$

From the recursive relation for $\langle \cdot \rangle^{\mathrm{W}}$, Eq. (I.1.4), or from the explicit formula (I.2.16), we find by means of Eq. (1.5)

$$\langle V(t_1) \cdots V(t_n)\, V(s) \rangle^{\mathrm{W}} = \nu \int d\tau\, \varphi(t_1; \tau) \cdots \varphi(s; \tau) + O(\nu^2) \qquad (1.8)$$

where $O(\nu^2)$ contains higher orders of ν. We insert this into Eq. (1.7). Then a partial summation of all terms proportional to ν can be performed,

$$\begin{aligned} G_{\mathrm{I}}(t, s) &= \sum_0^\infty \int_s^t dt_1 \cdots \int_s^{t_{n-1}} dt_n \\ &\quad \times \left[\nu \int d\tau\, \varphi(t_1; \tau) \cdots \varphi(t_n; \tau)\, \varphi(s; \tau) + O(\nu^2) \right] \\ &= \nu \int d\tau\, \mathscr{T} \exp\left[\int_s^t dt'\, \varphi(t'; \tau) \right] \varphi(s; \tau) + O(\nu^2) \end{aligned} \qquad (1.9)$$

Partial summation of the ν^n terms can also be performed in principle—although in Section 4 we will use a more direct method—to give a power series in ν,

$$G_{\mathrm{I}}(t, s) = \sum_1^\infty G_n(t, s)\, \nu^n \qquad (1.10)$$

where, by Eq. (1.9), G_1 is given by

$$G_1(t, s) = \int d\tau\, \mathscr{T} \exp\left[\int_s^t dt'\, \varphi(t'; \tau) \right] \varphi(s; \tau) \qquad (1.11)$$

Terminating the expansion of G_{I} at $n = 1, 2, \ldots$ will lead, via the integral equation (1.6), to different approximations for $\langle U(t, t_0) \rangle$.

In the stationary case a single pulse and G_{I} as well as G_n are of the form

$$\begin{gathered} \varphi(t; \tau) \equiv \varphi(t - \tau) \\ G_{\mathrm{I}}(t, s)) \equiv G_{\mathrm{I}}(t - s), \qquad G_n(t, s) \equiv G_n(t - s) \end{gathered} \qquad (1.12)$$

In this stationary case the integral equation (1.6) is immediately solved by Laplace transform, denoted by $\mathscr{L}_p$ and a caret:

$$\mathscr{L}_p\{\langle U(t, 0) \rangle\} = [p - p\hat{G}_{\mathrm{I}}]^{-1} \qquad (1.13)$$

where t is a symbolic variable only. For the first-order approximation to this we note that

$$p\hat{G}_1 - G_1(0) = \hat{\dot{G}}_1$$

and therefore, by Eq. (1.3),

$$pv\hat{G}_1 = \langle V \rangle + v\mathscr{L}_p \left\{ \int d\tau\, \varphi(t-\tau) \mathscr{T} \exp\left[\int_0^t dt'\, \varphi(t'-\tau) \right] \varphi(-\tau) \right\} \tag{1.14}$$

Inserting this for $p\hat{G}_{\text{I}}$ into Eq. (1.13) gives the first-order approximation. This is the same as the so-called "unified theory" of spectral-line broadening,[6] except for an averaging over additional parameters, as will be explained in more detail elsewhere.[4]

As regards singularities, G_1 and also G_n are much better behaved then the correlation functions of V. This is particularly evident if one writes G_1 of Eq. (1.11) as a derivative,

$$G_1(t, s) = -\frac{\partial}{\partial s} \int d\tau \left\{ \mathscr{T} \exp\left[\int_s^t dt'\, \varphi(t'; s) \right] - \mathbf{1} \right\} \tag{1.15}$$

Since for higher order corrections the partial summations become increasingly tedious, we will use another method in Section 4 to determine G_n for arbitrary n [cf. Eq. (4.16)]. The second order, G_2, is written out in Eq. (4.17) and it contains correlations from two pulses, while G_1 contains only a single pulse.

1.3. Density Expansion through a Differential Equation

A differential equation for $\langle U(t, t_0) \rangle$,

$$\frac{d}{dt} \langle U(t, t_0) \rangle = K(t, t_0) \langle U(t, t_0) \rangle \tag{1.16}$$

was studied by Kubo,[5] van Kampen,[7] and in I. The K is given by Eq. (I.1.7) in terms of K-cumulants,

$$\begin{aligned} K(t, t_0) &= \left\langle V(t) \mathscr{T} \exp\left[\int_{t_0}^t dt'\, V(t') \right] \right\rangle^{\text{K}} \\ &\equiv \sum_1^\infty \int_{t_0}^t dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n \, \langle V(t)\, V(t_1) \cdots V(t_n) \rangle^{\text{K}} \end{aligned} \tag{1.17}$$

From the recursive relation for $\langle \cdot \rangle^{\text{K}}$, Eq. (I.1.8), or from the explicit formula (I.3.9), we find by means of Eq. (1.5)

$$\langle V(t)\, V(t_1) \cdots V(t_n) \rangle^{\text{K}} = v \int d\tau\, \varphi(t; \tau) \cdots \varphi(t_n; \tau) + O(v^2) \tag{1.18}$$

Inserting this into Eq. (1.17), we can perform a partial summation of all terms proportional to ν to yield

$$K(t, t_0) = \nu \int d\tau\, \varphi(t; \tau)\, \mathscr{T} \exp\left[\int_{t_0}^{t} dt'\, \varphi(t'; \tau)\right] + O(\nu^2) \qquad (1.19)$$

Partial summation of the ν^n terms gives in principle a power series in ν,

$$K(t, t_0) = \sum_{1}^{\infty} K_n(t, t_0)\, \nu^n \qquad (1.20)$$

where, by Eq. (1.19),

$$\begin{aligned} K_1(t, t_0) &= \int_{-\infty}^{\infty} d\tau\, \varphi(t; \tau)\, \mathscr{T} \exp\left[\int_{t_0}^{t} dt'\, \varphi(t'; \tau)\right] \\ &= \frac{d}{dt} \int d\tau \left\{\mathscr{T} \exp\left[\int_{t_0}^{t} dt'\, \varphi(t'; \tau)\right] - 1\right\} \end{aligned} \qquad (1.21)$$

This gives the first-order approximation

$$\langle U(t, t_0)\rangle^{(1)} = \mathscr{T} \exp\left[\int_{t_0}^{t} dt'\, K_1(t', t_0)\right] \qquad (1.22)$$

For *commuting* single pulses this result is already the *exact* solution, since in this case we will show $K_n = 0$ for $n \geqslant 2$. Time-ordering can then be omitted in Eq. (1.22), which then agrees with the well-known expression for scalar shot noise.[2]

The general K_n will be determined explicitly by another method in Section 3. As regards singularities, the K_n are again better behaved than the correlation functions of V. The K_1 depends on a single pulse, while K_n depends on n pulses.

In the stationary case K_1 and G_1 are identical, as seen from Eqs. (1.21) and (1.15), but the corresponding approximations to $\langle U(t, 0)\rangle$ are in general different. For example, if $\varphi(t; \tau)$ is commutative, the first-order approximation to the integral equation does *not* give the exact solution. The particular situation dictates which of the two expansions is better adapted to the original problem. For example, if the Fourier transform of $\langle U(t, 0)\rangle$ is close to a Cauchy distribution, the integral equation and an approximation of its kernel G_1 might be more appropriate.

1.4. Overview of Remaining Sections. Generalizations

In Section 2 we establish the starting point for Sections 3 and 4. In addition, we derive a generalized Dyson series for $\langle U(t, 0)\rangle$, with τ as an auxiliary time variable. To this series the K-cumulant expansion for generalized Dyson series of I can be applied directly, giving yet another power series in ν. All three types of expansions are treated as formal power series. Convergence properties are not studied here. In Section 3 the explicit form of K_n is determined, and in Section 4 a recursion relation for G_n is derived, which is solved explicitly for the stationary case. In the Appendix the correlation functions of $V(t)$ are explicitly given.

In applications the single pulse φ may depend on additional variables, such as velocity or charge, over which one also has to average. If these additional random variables are independent of the arrival times, then all results can be caried over. The main change is that integration over τ is replaced by an integration over τ and the additional variables with appropriate weights. This will be exemplified for spectral-line broadening in a forthcoming paper.[4]

2. PREREQUISITES. GENERALIZED DYSON SERIES FOR QUANTUM SHOT NOISE

2.1. Preliminaries

For shot noise with pulse density ν and arrival times distributed uniformly over the real axis the probability for n particles to arrive in a time interval of length T is

$$p_n(T) = \frac{1}{n!}(\nu T)^n e^{-\nu T} \tag{2.1}$$

Let $\tau_1,..., \tau_n$ be arrival times and let

$$U(t, t_0; \tau_1,..., \tau_n) := \mathscr{T} \exp\left[\int_{t_0}^{t} dt' \sum_1^n \varphi(t'; \tau_i)\right] \tag{2.2}$$

Note the symmetry in $\tau_1,..., \tau_n$. We let $[\tau_0, \tau]$ be a large, but finite interval and denote by $\langle U(t, t_0) | \tau_0, \tau\rangle$ the expectation of the solution of Eq. (1.2) under the condition that the particles arrive in $[\tau_0, \tau]$. Then, with $T := \tau - \tau_0$ and by Eq. (2.1),

$$\begin{aligned}
&\langle U(t, t_0) | \tau_0, \tau\rangle \\
&= e^{-\nu T}\left[1 + \sum_1^\infty \frac{1}{n!}(\nu T)^n e^{-\nu T} \int_{\tau_0}^{\tau} \frac{d\tau_1}{T} \cdots \int_{\tau_0}^{\tau} \frac{d\tau_n}{T} U(t, t_0; \tau_1,..., \tau_n)\right] \\
&= e^{-\nu T}\left[1 + \sum_1^\infty \frac{\nu^n}{n!} \int_{[\tau_0, \tau]^n} d^n\tau\, U(t, t_0; \tau_1,..., \tau_n)\right]
\end{aligned} \tag{2.3}$$

We are interested in the limit $[\tau, \tau] \to \mathbb{R}$, $T \to \infty$. This limit cannot be directly performed, since $\exp(-\nu T) \to 0$ and the integrals diverge. Nevertheless, Eq. (2.3) can be used to determine $K_n(t, t_0)$ in a neat way, as shown in Section 3. For the determination of $G_n(t, s)$, however, we need the explicit limit $T \to \infty$, and we show in Section 4 that with a trick it can indeed be performed. The remaining part of Section 2 is a special case of results in ref. 8. It is not needed for the rest of the paper, but is of independent interest.

2.2. Cumulant Expansion for Generalized Dyson Series

By symmetry, the integrals in Eq. (2.3) can be restricted to time-ordered domains,

$$\begin{aligned} &\langle U(t, t_0) \mid \tau_0, \tau \rangle \\ &= e^{-\nu T} \left[1 + \sum_1^\infty \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \cdots \int_{\tau_0}^{\tau_{n-1}} d\tau_n \; \nu^n U(t, t_0; \tau_1, \ldots, \tau_n) \right] \end{aligned} \tag{2.4}$$

We now treat t and t_0 as fixed parameters and τ as (auxiliary) time. Then, by Eq. (2.4),

$$F(\tau, \tau_0) := e^{\nu(\tau - \tau_0)} \langle U(t, t_0) \mid \tau_0, \tau \rangle \tag{2.5}$$

is a generalized Dyson series to which we can apply the K-cumulant expansion for generalized Dyson series of I. Hence, by Eqs. (I.1.14) and (I.1.15), F satisfies

$$\frac{d}{d\tau} F(\tau, \tau_0) = K_{\mathrm{Dy}}(\tau, \tau_0)\, F(\tau, \tau_0) \tag{2.6}$$

where K_{Dy} is a power series in ν,

$$\begin{aligned} K_{\mathrm{Dy}}(\tau, \tau_0) = \nu U(t, t_0; \tau) + \sum_1^\infty \nu^{n+1} \int_{\tau_0}^{\tau} d\tau_1 \cdots \int_{\tau_0}^{\tau_{n-1}} d\tau_n \\ \times U^{\mathrm{K}}(t, t_0; \tau, \tau_1, \ldots, \tau_n) \end{aligned} \tag{2.7}$$

Here the U^{K} are K-cumulants with respect to $\tau, \tau_1, \ldots, \tau_n$. From Eq. (2.6) one obtains

$$\frac{\partial}{\partial\tau} \langle U(t, t_0) \mid \tau_0, \tau \rangle = (K_{\mathrm{Dy}} - \nu) \langle U(t, t_0) \mid \tau_0, \tau \rangle \tag{2.8}$$

The initial condition is

$$\langle U(t, t_0) | \tau_0, \tau_0 \rangle = 1$$

and the solution of Eq. (2.8) is a τ-time-ordered exponential,

$$\langle U(t, t_0) | \tau_0, \tau \rangle = \mathscr{T}_{\tau'} \exp \left\{ \nu \int_{\tau_0}^{\tau} d\tau' \, [K_{\mathrm{Dy}} - \nu] \right\} \tag{2.9}$$

Now the limit $-\tau_0, \tau \to \infty$ can be performed and one obtains $\langle U(t, t_0) \rangle$. Retaining only terms proportional to ν in K_{Dy}, one obtains the approximation

$$\langle U(t, t_0) \rangle^{(1)} = \mathscr{T}_{\tau} \exp \left\{ \nu \int_{-\infty}^{\infty} d\tau \, [U(t, t_0; \tau) - 1] \right\} \tag{2.10}$$

This will be finite if, for $|\tau| \to \infty$, one has $\varphi(t; \tau) \to 0$ sufficiently fast so that the integral in Eq. (2.10) exists.

For *commuting* single pulses one has that $U(t, t_0; \tau_1, \ldots, \tau_n)$ factorizes, and then $U^{\mathrm{K}} \equiv 0$ for $n \geqslant 2$, by the cluster property. In the commuting case the first order is already the *exact* solution. One can then omit the time-ordering $\mathscr{T}_{\tau}$ in Eq. (2.10), and the result agrees with Eq. (1.22) and with the well-known result for the scalar case.[2]

For the general case the second-order approximation is obtained by retaining only ν and ν^2 terms in K_{Dy},

$$\begin{aligned} &\langle U(t, t_0) \rangle^{(2)} \\ &\quad = \mathscr{T}_{\tau_1} \exp \left\{ \int_{-\infty}^{\infty} d\tau_1 \left[\nu (U(t, t_0; \tau_1) - 1) + \nu^2 \int_{-\infty}^{\tau_1} d\tau_2 \, U^{\mathrm{K}}(t, t_0; \tau_1, \tau_2) \right] \right\} \end{aligned} \tag{2.11}$$

where

$$U^{\mathrm{K}}(t, t_0; \tau_1, \tau_2) = U(t, t_0; \tau_1, \tau_2) - U(t, t_0; \tau_1) \, U(t, t_0; \tau_2) \tag{2.12}$$

3. EXPANSION BASED ON A DIFFERENTIAL EQUATION

In this section we explicitly determine each term $K_n(t, t_0)$ in the expansion of $K(t, t_0)$ in Eq. (1.20). To this end, we consider a finite interval $[\tau_0, \tau]$ of arrival times and study the *ansatz*

$$\frac{d}{dt} \langle U(t, t_0) | \tau_0, \tau \rangle = K(t, t_0; \tau_0, \tau) \langle U(t, t_0) | \tau_0, \tau \rangle \tag{3.1}$$

where $\langle U(t, t_0) | \tau_0 \tau \rangle$ is given by Eq. (2.3). Later we let $-\tau_0, \tau \to \infty$. Using Eq. (2.2), we define

$$U_n := \frac{1}{n!} \int_{[\tau_0, \tau]^n} d^n\boldsymbol{\tau}\; U(t, t_0; \tau_1, ..., \tau_n) \tag{3.2}$$

We suppress the dependence of K on t, t_0, τ_0, τ and consider an expansion analogous to Eq. (1.20),

$$K = \sum_1^\infty K_n \nu^n \tag{3.3}$$

Inserting Eq. (2.3) into Eq. (3.1) gives, with Eqs. (3.2) and (3.3),

$$\sum_1^\infty \nu^n \dot{U}_n = \left(\sum_1^\infty \nu^\alpha K_\alpha \right) \left(1 + \sum_1^\infty \nu^\beta U_\beta \right) \tag{3.4}$$

Comparing equal powers of ν, one obtains

$$\dot{U}_n = K_n + \sum_{j=1}^{n-1} K_j U_{n-j} \tag{3.5}$$

This is a recursive relation for K_n. One finds for $n = 1$ and $n = 2$

$$K_1 = \dot{U}_1 = \int_{\tau_0}^{\tau} d\tau_1\; \varphi(t; \tau_1)\; U(t, t_0; \tau_1) \tag{3.6}$$

$$\begin{aligned} K_2 &= \dot{U}_2 - \dot{U}_1 U_1 \\ &= \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau} d\tau_2 \left\{ \frac{1}{2!} [\varphi(t; \tau_1) + \varphi(t; \tau_2)]\; U(t, t_0; \tau_1, \tau_2) \right. \\ &\quad \left. - \varphi(t; \tau_1)\; U(t, t_0; \tau_1)\; U(t, t_0; \tau_2) \right\} \end{aligned}$$

A change of variables $\varphi(t; \tau_2) \to \varphi(t; \tau_1)$ and the symmetry of $U(t, t_0; \tau_1, \tau_2)$ yields

$$\begin{aligned} K_2 = &\int_{\tau}^{\tau} d\tau_1 \int_{\tau_0}^{\tau} d\tau_2\; \varphi(t; \tau_1) \\ &\times [U(t, t_0; \tau_1, \tau_2) - U(t, t_0; \tau_1)\; U(t, t_0; \tau_2)] \end{aligned} \tag{3.7}$$

For K_3 one can proceed similarly, and this suggests as general solution

$$K_n = \frac{1}{(n-1)!} \int_{[\tau_0, \tau]^n} d^n\boldsymbol{\tau}\; \varphi(t; \tau_1)\; U^{\mathrm{K}}(t, t_0; \tau_1, ..., \tau_n) \tag{3.8}$$

where the U^{K} are K-cumulants of $U(t, t_0; \tau_1, ..., \tau_n)$ with respect to $\tau_1, ..., \tau_n$. Proof is by induction. Assume this to hold for $n-1$. Then, from Eq. (3.5) and by a change of variables in the time-derivative term as before, one obtains

$$K_n = \int_{[\tau_0, \tau]^n} d^n \boldsymbol{\tau} \left[\frac{1}{(n-1)!} \varphi(t; \tau_1)\, U(t, t_0; \tau_1, ..., \tau_n) \right.$$
$$- \varphi(t; \tau_1) \sum_{\alpha=1}^{n-1} \frac{1}{(\alpha-1)!} U^{\mathrm{K}}(t, t_0; \tau_1, ..., \tau_\alpha)$$
$$\left. \times \frac{1}{(n-\alpha)!} U(t, t_0; \tau_{\alpha+1}, ..., \tau_n) \right]$$
$$= \frac{1}{(n-1)!} \int_{[\tau_0, \tau]^n} d^n \boldsymbol{\tau}\, \varphi(t; \tau_1) \left[U(t, t_0; \tau_1, ..., \tau_n) \right.$$
$$\left. - \sum_{\alpha=1}^{n-1} \binom{n-1}{\alpha-1} U^{\mathrm{K}}(t, t_0; \tau_1, ..., \tau_\alpha)\, U(t, t_0; \tau_{\alpha+1}, ..., \tau_n) \right] \tag{3.9}$$

Now we use as a *crucial* fact that U^{K} is symmetric in $\tau_2, ..., \tau_n$, $n \geqslant 2$, i.e.,

$$U^{\mathrm{K}}(t, t_0; \tau_1, ..., \tau_n) = U^{\mathrm{K}}(t, t_0; \tau_1, \tau_{\pi(2)}, ..., \tau_{\pi(n)}) \tag{3.10}$$

for any permutation of $(2, ..., n)$. This follows from the symmetry of $U(t, t_0; \tau_1, ..., \tau_n)$ and from the recursive relation for U^{K} in Eq. (1.8) of I by induction.

For $\Lambda = \{\lambda_1, ..., \lambda_m\}$, $\lambda_1 < \cdots < \lambda_m$, we denote

$$\boldsymbol{\tau}_\Lambda = \{\tau_{\lambda_1}, ..., \tau_{\lambda_m}\}$$

There are $\binom{n-1}{m}$ different ways to select m numbers from $\{2, ..., n\}$. By symmetry we can therefore write Eq. (3.9) as

$$K_n = \frac{1}{(n-1)!} \int_{[\tau_0, \tau]^n} d^n \boldsymbol{\tau}\, \varphi(t; \tau_1) \left[U(t, t_0; \tau_1, ..., \tau_n) \right.$$
$$\left. - \sum_{\Lambda_1 \cup \Lambda_2 = \{2, ..., n\}} U^{\mathrm{K}}(t, t_0; \tau_1, \boldsymbol{\tau}_{\Lambda_1})\, U(t, t_0; \boldsymbol{\tau}_{\Lambda_2}) \right]$$

where $\Lambda_2 \neq \varnothing$. The square brackets just give $U^{\mathrm{K}}(t, t_0; \tau_1, ..., \tau_n)$, by Eq. (I.1.8). This proves Eq. (3.8).

Now it is possible to perform the limit $\tau_0 \to -\infty$, $\tau \to \infty$ in Eq. (3.1). If, for fixed t, $\varphi(t; \tau)$ vanishes outside a finite τ interval or if it approaches zero sufficiently rapidly, then $\varphi(t; \tau_1)\, U^{\mathrm{K}}(t, t_0; \tau_1, ..., \tau_n)$ will be integrable. For $n = 1$ this is evident and for $n \geqslant 2$ this follows from the cluster property.

Indeed, if $\tau_i \to \infty$, $i \geqslant 2$, we can assume $i = n$, by symmetry, Eq. (3.10). The $U(t, t_0; \tau_n) \to \mathbf{1}$ and

$$U(t, t_0; \tau_1, ..., \tau_n) \to U(t, t_0; \tau_1, ..., \tau_{n-1})\, U(t, t_0; \tau_n)$$

Hence, $U^{\mathrm{K}} \to 0$ for $n \geqslant 2$. For $[\tau_0, \tau] \to \mathbb{R}$, $K(t, t_0; \tau_0, \tau)$ becomes

$$K(t, t_0) = \sum_{n=1}^{\infty} v^n \frac{1}{(n-1)!} \int_{\mathbb{R}^n} d^n\boldsymbol{\tau}\, \varphi(t, \tau_1)\, U(t, t_0; \tau_1, ..., \tau_n) \tag{3.11}$$

and Eq. (3.1) becomes

$$\frac{d}{dt} \langle U(t, t_0) \rangle = K(t, t_0) \langle U(t, t_0) \rangle \tag{3.12}$$

In order for this to hold, one only needs that the left-hand side of Eq. (3.1) converges in the limit $[\tau_0, \tau] \to \mathbb{R}$.

From the uniqueness of $K(t, t_0)$ and of its expansion in powers of v it follows that $v^n K_n$ coincides with the partial sums of Section 1.

In the case of *commuting* single pulses, $U(t, t_0; \tau_1, ..., \tau_n)$ factorizes into $\prod U(t, t_0; \tau_i)$, so that one has $U^{\mathrm{K}} \equiv 0$ or $n \geqslant 2$, and therefore the first-order approximation already gives the exact result. It then agrees with Eq. (2.10) and with the classical result for the scalar case.[2]

4. EXPANSION OF THE INTEGRAL KERNEL

We now derive an alternative density expansion for quantum shot noise, of which Eq. (1.11) is the first term. To this end, we return to the expression for $\langle U(t, t_0) | \tau_0 \tau \rangle$ in Eq. (2.3). The difficulty is that the limit $T \to \infty$ cannot be performed directly, since then the integrals diverge and $\exp(-vT) \to 0$. To obtain a better behaved expression, we also expand $\exp(-vT)$ into a power series and collect equal powers of v. This gives

$$\begin{aligned} \langle U(t, t_0) | \tau_0, \tau \rangle &= \sum_{n=0}^{\infty} \sum_{\alpha+\beta=n} \frac{1}{\alpha!\, \beta!} (-1)^{\alpha} T^{\alpha} v^{\alpha+\beta} \\ &\quad \times \int_{[\tau_0 \tau]^{\beta}} d^{\beta}\boldsymbol{\tau}\, U(t, t_0; \tau_1, ..., \tau_{\beta}) \\ &= \sum_{n=0}^{\infty} v^n \frac{1}{n!} \int_{[\tau_0, \tau]^n} d^n\boldsymbol{\tau} \\ &\quad \times \sum_{\beta=0}^{n} \frac{n!}{\beta!\, (n-\beta)!} (-1)^{n-\beta}\, U(t, t_0; \tau_1, ..., \tau_{\beta}) \end{aligned} \tag{4.1}$$

where T^α has been replaced by a trivial α-fold integral. We now symmetrize the integrand in Eq. (4.1). Since there are $\binom{n}{\beta}$ ways to select β different numbers from $1,\dots, n$, the *symmetrized integrand*, which we denote by U^Δ, becomes

$$U^\Delta(t, t_0; \tau_1,\dots, \tau_n) = \sum_{\Lambda \subset \{1,\dots, n\}} (-1)^{n-|\Lambda|}\, U(t, t_0; \boldsymbol{\tau}_\Lambda) \tag{4.2}$$

where for the empty subset we set $U(\tau_\varnothing) := 1$ and where $|\Lambda|$ denotes the number of elements in the set Λ. In particular,

$$\begin{aligned} U^\Delta(\tau_1) &= U(\tau_1) - 1 \\ U^\Delta(\tau_1, \tau_2) &= U(\tau_1, \tau_2) - U(\tau_1) - U(\tau_2) + 1 \end{aligned} \tag{4.3}$$

If the single-pulse shape $\varphi(t; \tau)$ vanishes for $\tau \to \pm\infty$, then

$$U^\Delta(t, t_0; \tau_1,\dots, \tau_n) \to 0 \qquad \text{as} \quad \tau_i \to \pm\infty \tag{4.4}$$

This is evident in Eq. (4.3) and it follows for the general case by induction based on the formula

$$U(\tau_1,\dots, \tau_n) = \sum_{\Lambda \subset \{1,\dots, n\}} U^\Delta(\boldsymbol{\tau}_\Lambda) \tag{4.5}$$

obtained from Eq. (4.3) by summation. See also ref. 9. By Eq. (4.4), the limit τ_0, $\tau \to \pm\infty$ in

$$\langle U(t, t_0) \mid \tau_0, \tau \rangle = \sum_0^\infty \frac{v^n}{n!} \int_{[\tau_0, \tau]^n} d^n\boldsymbol{\tau}\; U^\Delta(t, t_0; \tau_1,\dots, \tau_n) \tag{4.6}$$

can now be taken explicitly if the single-pulse shape vanishes sufficiently rapidly. We set

$$U_n^\Delta(t, t_0) := \frac{1}{n!} \int_{\mathbb{R}^n} d^n\boldsymbol{\tau}\; U^\Delta(t, t_0; \tau_1,\dots, \tau_n) \tag{4.7}$$

Then Eq. (4.6) becomes

$$\langle U(t, t_0) \rangle = \sum v^n U_n^\Delta(t, t_0) \tag{4.8}$$

Now it is possible to determine the expansion of the integral kernel G_I. By Eqs. (1.6)–(1.10), the lhs of Eq. (4.8) equals

$$1 + \int_{t_0}^t ds \sum_{\alpha=1}^\infty \sum_{\beta=0}^\infty v^\alpha G_\alpha(t, s)\, v^\beta U_n^\Delta(s, t_0)$$

Comparing equal powers of v, we obtain a recursion relation for G_n,

$$U_n^{\Delta}(t, t_0) = \int_{t_0}^{t} ds \left[G_n(t, s) + \sum_{\alpha=1}^{n-1} G_\alpha(t, s)\, U_{n-\alpha}^{\Delta}(s, t_0) \right] \tag{4.9}$$

This is simplified by introducing

$$H_n(t, s) := \int_s^t ds'\, G_n(t, s') \tag{4.10}$$

so that

$$G_n(t, s) = -\frac{\partial}{\partial s} H_n(t, s) \tag{4.11}$$

By partial integration, Eq. (4.9) then becomes a uniquely solvable recursion relation for H_n,

$$H_n(t, t_0) = U_n^{\Delta}(t, t_0) - \sum_{\alpha=1}^{n-1} \int_{t_0}^{t} ds\, H_\alpha(t, s) \frac{\partial}{\partial s} U_{n-\alpha}^{\Delta}(s, t_0) \tag{4.12}$$

For $n = 1$ we recover Eq. (1.11),

$$\begin{aligned} H_1(t, t_0) &= U_1^{\Delta}(t, t_0) = \int d\tau\, [U(t, t_0; \tau) - 1] \\ G_1(t, s) &= \int d\tau\, U(t, s; \tau)\, \varphi(s; \tau) \end{aligned} \tag{4.13}$$

We will give a closed expression for G_n in the stationary case, which is of main interest for applications.

The Stationary Case. If the single-pulse contribution depends on the time difference only,

$$\varphi(t; \tau) = \varphi(t - \tau) \tag{4.14}$$

then the process $V(t)$ is stationary, and

$$G_n(t, s) = G_n(t - s)$$

since the same holds for G_{I}. Now, Eq. (4.9) is easily solved in terms of Laplace transforms,

$$\hat{G}_n(p) = \int_0^\infty dt\, e^{-pt}\, G_n(t)$$

Putting $t_0 = 0$ and Laplace-transforming Eq. (4.9), we obtain

$$\hat{U}_n^{\Delta} = \frac{1}{p}\,\hat{G}_n + \sum_{\alpha=1}^{n-1} \hat{G}_\alpha \hat{U}_{n-\alpha}^{\Delta}$$

Since $U_n^{\Delta}(0, 0) = 0$, this can be written as

$$\hat{U}_n^{\Delta} = \hat{G}_n + \sum_{\alpha=1}^{n-1} \hat{G}_\alpha \hat{U}_{n-\alpha}^{\Delta} \tag{4.15}$$

The solution is immediately seen to be[3]

$$\hat{G}_n = \sum_{n_1 + \cdots n_k = n} (-1)^{k-1}\, \hat{U}_{n_1}^{\Delta} \cdots \hat{U}_{n_k}^{\Delta} \tag{4.16}$$

where $n_i \geqslant 1$. For $n = 1$ we obtain

$$\hat{G}_1 = \mathscr{L}_p \left\{ \int d\tau\, \varphi(t-\tau)\, U(t, 0; \tau) \right\} \tag{4.17}$$

from which Eq. (1.14) follows. For $n = 2$ we obtain, with Eqs. (4.3) and (4.7),

$$\hat{G}_2 = \mathscr{L}_p \left(\frac{1}{2} \int d^2\boldsymbol{\tau}\, \{ [\varphi(t-\tau_1) + \varphi(t-\tau_2)]\, U(t, 0; \tau_1, \tau_2) - \varphi(t-\tau_1)\, U(t, 0; \tau_1) - \varphi(t-\tau_2)\, U(t, 0; \tau_2) \} \right) - \hat{G}_1^2 \tag{4.18}$$

By a change of variables this can be written as

$$\hat{G}_2 = \mathscr{L}_p \left\{ \int d^2\boldsymbol{\tau}\, \varphi(t-\tau_1)\, [U(t, 0; \tau_1, \tau_2) - U(t, 0; \tau_1)] \right\} - \hat{G}_1^2 \tag{4.19}$$

In the limiting case of δ-like pulses, the first-order approximation to G_{I} becomes *exact*, also in the nonstationary case. For

$$\varphi(t; \tau) = \phi(\tau)\, \delta(t-\tau) \tag{4.20}$$

one can show by means of Eq. (4.9) that

$$G_{\mathrm{I}}(t, s) = v G_1(t, s) = v[e^{\phi(s)} - \mathbf{1}] \tag{4.21}$$

[3] It is interesting to note a connection of Eq. (4.15) with W-cumulants. If we set $\hat{U}^{\Delta}(\alpha+1, \dots, n) := \hat{U}_{n-\alpha}^{\Delta}$ then Eq. (4.15) becomes identical to the recursion relation (I.1.4) for W-cumulants, so that $\hat{G}_n = (\hat{U}^{\Delta}(1, \dots, n)^{\mathrm{W}}$. From this, Eq. (4.16) also follows, by Eq. (I.21.6).

The integral equation then gives

$$\langle U(t, t_0)\rangle = \mathscr{T} \exp\left\{ v \int_{t_0}^{t} dt' \, [e^{\phi(t')} - \mathbf{1}] \right\} \tag{4.22}$$

For stationary δ-pulses one has commutativity and a special scalar case. We note that $e^{\phi(\tau)}$ is a sort of "S-matrix" for a pulse occurring at time τ.

APPENDIX. CALCULATION OF $\langle V(t_1) \cdots V(t_n)\rangle$ FOR QUANTUM SHOT NOISE

The correlation functions can be obtained by specializing a more general method of ref. 10. It is, however, instructive to use a more pedestrian approach, such as the following. We consider N arrival times, which are uniformly distributed in a finite interval $[-T/2, T/2]$ with fixed $N/T = v$. Eventually, we will let $N, T \to \infty$. Writing $\varphi_i(t) \equiv \varphi(t; \tau_i)$ one has

$$V(t) = \sum_{1}^{N} \varphi_i(t)$$

As an example, we first calculate

$$\langle V(t_1) \, V(t_2) \, V(t_3)\rangle = \sum_{i_1, i_2, i_3} \langle \varphi_{i_1}(t_1) \, \varphi_{i_2}(t_2) \, \varphi_{i_3}(t_3)\rangle \tag{A.1}$$

This is decomposed into terms according to the number of different arrival times, leading to

$$\begin{aligned}
&\sum_{i_1} \langle \varphi_{i_1}(t_1) \, \varphi_{i_1}(t_2) \, \varphi_{i_1}(t_3)\rangle \\
&\qquad + \sum_{i_1 \neq i_2} \langle \varphi_{i_1}(t_1) \, \varphi_{i_1}(t_2) \, \varphi_{i_2}(t_3) \\
&\qquad + \varphi_{i_1}(t_1) \, \varphi_{i_2}(t_2) \, \varphi_{i_1}(t_3) + \varphi_{i_1}(t_1) \, \varphi_{i_2}(t_2) \, \varphi_{i_2}(t_3)\rangle \\
&\qquad + \sum_{i_k \neq i_l} \langle \varphi_{i_1}(t_1) \, \varphi_{i_2}(t_2) \, \varphi_{i_3}(t_3)\rangle
\end{aligned} \tag{A.2}$$

The numbers of $\langle \cdot \rangle$ terms in the sums are N, $N(N-1)$, and $N(N-1)(N-2)$, respectively, and each summand is independent of the particular value of i_1, i_2, i_3. Hence,

$$\begin{aligned}
&\langle V(t_1)\, V(t_2)\, V(t_3)\rangle \\
&= N \int_{-T/2}^{T/2} \frac{d\tau_1}{T}\, \varphi(t_i; \tau_1)\, \varphi(t_2; \tau_1)\, \varphi(t_3; \tau_1) \\
&\quad + N(N-1) \int_{-T/2}^{T/2} \frac{d\tau_1}{T} \int_{T/2}^{T/2} \frac{d\tau_2}{T} \{\varphi(t_1; \tau_1)\, \varphi(t_2; \tau_1)\, \varphi(t_3; \tau_2) \\
&\quad + \varphi(t_1; \tau_1)\, \varphi(t_2; \tau_2)\, \varphi(t_3; \tau_1) + \varphi(t_1; \tau_1)\, \varphi(t_2; \tau_2)\, \varphi(t_3; \tau_2)\} \\
&\quad + N(N-1)(N-2) \int_{[-T/2, T/2]^3} \prod_{\alpha=1}^{3} \frac{d\tau_\alpha}{T}\, \varphi(t_\alpha; \tau_\alpha) \qquad \text{(A.3)}
\end{aligned}$$

In the limit $T \to \infty$ this becomes

$$\begin{aligned}
&\langle V(t_1)\, V(t_2)\, V(t_3)\rangle \\
&= \nu \int d\tau \prod_{\alpha=1}^{3} \varphi(t_\alpha; \tau) \\
&\quad + \nu^2 \int d^2\tau\, \varphi(t_1; \tau_1) [\varphi(t_2; \tau_1)\, \varphi(t_3; \tau_2) \\
&\quad + \varphi(t_2; \tau_2)\, \varphi(t_3; \tau_1) + \varphi(t_2; \tau_2)\, \varphi(t_3; \tau_2)] \\
&\quad + \nu^3 \prod_{\alpha=1}^{3} \int d\tau_\alpha\, \varphi(t_\alpha, \tau_\alpha) \qquad \text{(A.4)}
\end{aligned}$$

For later applications to the general case we return to Eq. (A.2) and symmetrize its summands $\langle \cdot \rangle$ by permutations of the indices. The result can be written as

$$\sum_{m=1}^{3} \sum_{i_k \neq i_l} \frac{1}{m!} \sum_{\bigcup_{k=1}^{3} \{\alpha_j\} = \{1,\ldots,m\}} \langle \varphi_{i_{\alpha_1}}(t_1)\, \varphi_{i_{\alpha_2}}(t_2)\, \varphi_{i_{\alpha_3}}(t_2)\rangle \qquad \text{(A.5)}$$

Arguing as before, this yields in the limit $T \to \infty$

$$\begin{aligned}
&\langle V(t_1)\, V(t_2)\, V(t_3)\rangle \\
&= \sum_{m=1}^{3} \nu^m \int d^m\tau \frac{1}{m!} \sum_{\bigcup_{j=1}^{3} \{\alpha_j\} = \{1,\ldots,m\}} \varphi(t_1; \tau_{\alpha_1})\, \varphi(t_2; \tau_{\alpha_2})\, \varphi(t_3; \tau_{\alpha_3}) \qquad \text{(A.6)}
\end{aligned}$$

which is Eq. (A.4) with integrands symmetrized.

For $\langle V(t_1) \cdots V(t_n)\rangle$ we proceed similarly. The analog of Eq. (A.1) is again decomposed into sums according to the number of different arrival times, and their summands are symmetrized by permutations of the indices. This yields as an analog of Eq. (A.5)

$$\langle V(t_1)\cdots V(t_n)\rangle = \sum_{m=1}^{n} v^m \sum_{i_k\neq i_l} \frac{1}{m!} \sum_{\bigcup_{j=1}^{n}\{\alpha_j\}=\{1,\dots,m\}} \langle \varphi_{i_{\alpha_1}}(t_1)\cdots\varphi_{i_{\alpha_n}}(t_n)\rangle \tag{A.7}$$

which can be proved in this symmetrized form by induction. For fixed m the number of summands $\langle\cdot\rangle$ in each $\sum_{i_k\neq i_l}$ is $N(N-1)\cdots(N-m+1)$, and each summand is independent of the particular value of $i_1,\dots, i_m$. This gives an analog of Eq. (A.3) for finite T, and in the limit $T\to\infty$ one obtains

$$\langle V(t_1)\cdots V(t_n)\rangle = \sum_{m=1}^{n} v^m \int d^m\tau \frac{1}{m!} \sum_{\bigcup_{j=1}^{n}\{\alpha_j\}=\{1,\dots,m\}} \varphi(t_1;\tau_{\alpha_1})\cdots\varphi(t_n;\tau_{\alpha_n}) \tag{A.8}$$

The number of terms in each integrand can be reduced, if one wishes, by redoing the symmetrization. Identifying distributions of indices $\alpha_1,\dots,\alpha_m$ that differ only by a permutation, one can get rid of the $m!$ in Eq. (A.8) and obtain

$$\langle V(t_1)\cdots V(t_n)\rangle = \sum_{m=1}^{n} v^m \int d^m\tau \sum_{\substack{\bigcup_{j=1}^{n}\{\alpha_j\}=\{1,\dots,m\}\\ \text{mod perm.}}} \varphi(t_1;\tau_{\alpha_1})\cdots\varphi(t_n;\tau_{\alpha_n}) \tag{A.9}$$

For $n=3$ this reduces again to Eq. (A.4).

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